مفردات الفصل الدراسي الثاني لمادة التفاضل المتقدم ٢

Functions of several variables (Domain and range)
Graphs in 3D and level curves
Limits and Continues
Partial derivatives
Partial derivatives of higher order
The chain rule
Applications of Partial derivatives
Double integral
Examples of Reversing in double integral
Areas by double integral
Double integrals in Polar form
Tribe Integrals in rectangular coordinates
Tribe Integrals in cylindrical coordinates
Tribe Integrals in Spherical coordinates
Line integrals and Green 's Theorem
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Lecture Note

Advance Calculus

By

Mimoon Ibrahim Ismael

AL-ANBAR, IRAQ

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Contents

Chapter 1

Partial Differentiation

1.1 Functions of several variables

In real life, there are many formulas that depend on more than one variable. For example: Area of a rectangular $A = xy$, so A is a function of the two variables x and y. So, if $Z = f(x, y)$, then Z is a function of two variables x and y. Similarly, $W = f(x, y, z)$, a function of variables x, y and z.
Also, $U = f(x_1, x_2, \ldots, x_n)$, a function of variables $x_1, x_2, \ldots, x_n$.

Example: Evaluating the following functions?

- $h(x, y, z) = \ln(x^2 + y + z^2)$ at the point (-1,2,1).
- $g(r, s, t) = \sqrt{r^2 + s^2 + t^2}$ at the point (3,0,4).
- $f(x, y, z) = e^{\frac{(x+y)}{z}}$ at the point $(\ln 2, \ln 4, 3)$.
- $T(r, \theta) = \cos(\sqrt{r^2\theta^2 - 1})$ at the point (-1,-1).

Solution:

- The value of $h(x, y, z) = \ln(x^2 + y + z^2)$ at the point (-1,2,1) is

  $h(-1, 2, 1) = \ln\Big((-1)^2 + 2 + (1)^2\Big) = \ln 4.$

- The value of $g(r, s, t) = \sqrt{r^2 + s^2 + t^2}$ at the point (3,0,4) is

  $g(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = 5.$

- The value of $f(x, y, z) = e^{\frac{(x+y)}{z}}$ at the point $(\ln 2, \ln 4, 3)$ is

  $f(\ln 2, \ln 4, 3) = e^{\frac{(\ln 2+\ln 4)}{3}} = e^{\frac{\ln 8}{3}} = e^{\frac{3\ln 2}{3}} = 2.$

- The value of $T(r, \theta) = \cos(\sqrt{r^2\theta^2 - 1})$ at the point (-1,-1) is

  $T(-1, -1) = \cos\left(\sqrt{(-1)^2(-1)^2 - 1}\right) = \cos(0) = 1.$

1.2 Limit of a Function of Two Variables

We say that a function $f(x, y)$ approaches the limit a as (x, y) approaches (x_0, y_0) and write

$$\lim_{(x,y)\longrightarrow(x_0,y_0)} f(x, y) = a.$$

Note the following rules hold if $a, b \in \mathbb{R}$ and

$$\lim_{(x,y)\longrightarrow(x_0,y_0)} f(x, y) = a \text{ and } \lim_{(x,y)\longrightarrow(x_0,y_0)} g(x, y) = b$$

1.
$$\lim_{(x,y)\longrightarrow(x_0,y_0)} f(x, y) \pm \lim_{(x,y)\longrightarrow(x_0,y_0)} g(x, y) = a \pm b$$

2.
$$\lim_{(x,y)\longrightarrow(x_0,y_0)} f(x, y)g(x, y) = ab$$

3.
$$\lim_{(x,y)\longrightarrow(x_0,y_0)} \frac{f(x, y)}{g(x, y)} = \frac{a}{b}, \text{ where } b \neq 0$$

4.
$$\lim_{(x,y)\longrightarrow(x_0,y_0)} \Big[f(x, y)\Big]^n = a^n$$

Example: Find the limits of the functions if possible ?

1.
$$\lim_{(x,y)\longrightarrow(\sqrt{2},0)} \left(x^2 + xy\right)$$

2.
$$\lim_{(x,y)\longrightarrow(0,0)} \left(\sin(xy) - \cos(y^2)\right)$$

3.
$$\lim_{(x,y)\longrightarrow(2,-1)} \left[\frac{x+y}{x-y}\right]^3$$

4.
$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$$

Solution

1.
$$\lim_{(x,y)\longrightarrow(\sqrt{2},0)} x^2 + \lim_{(x,y)\longrightarrow(\sqrt{2},0)} xy = (\sqrt{2})^2 + (\sqrt{2})(0) = 2 + 0 = 2$$

2.
$$\lim_{(x,y)\longrightarrow(0,0)} \sin(xy) - \lim_{(x,y)\longrightarrow(0,0)} \cos(y^2) = \sin(0) - \cos(0) = -1$$

3.
$$\left[\frac{\lim_{(x,y)\longrightarrow(2,-1)} x+y}{\lim_{(x,y)\longrightarrow(2,-1)} x-y}\right]^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

4.
$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\longrightarrow(0,0)} \frac{x(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} = 0(\sqrt{0}+\sqrt{0}) = 0$$

1.3 Partial Differentiation

Consider the function $z = f(x,y)$, and let x change to $x + \Delta x$, while y remains constant. In this case, z will change to $z + \Delta z$, so that

$$\begin{aligned} z + \Delta z &= f(x+\Delta x, y), \\ \Delta z &= f(x+\Delta x, y) - z, \\ \Delta z &= f(x+\Delta x, y) - f(x,y), \\ \frac{\Delta z}{\Delta x} &= \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}. \end{aligned}$$

Now, upon taking the limit as Δx goes to zero, we have the partial derivative of z with respect to x:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\longrightarrow 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

Note that the partial derivative can be denoted by either

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } Z_x \text{ or } f_x(x,y).$$

Similarly, we have for $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \longrightarrow 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$

Note that the partial derivative can be denoted by either

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } Z_y \text{ or } f_y(x,y).$$

Example: Using the limit definition of partial derivative to find $\frac{\partial z}{\partial x}$ of the function $f(x,y) = xy$? Solution

$$\begin{aligned}\frac{\partial z}{\partial x} &= \lim_{\Delta x \longrightarrow 0} \frac{(x+\Delta x)y - xy}{\Delta x},\\ &= \lim_{\Delta x \longrightarrow 0} \frac{xy + y\Delta x - xy}{\Delta x},\\ &= \lim_{\Delta x \longrightarrow 0} \frac{y\Delta x}{\Delta x},\\ &= y.\end{aligned}$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

a- $f(x,y) = x^2 + 3xy + y - 1$

b- $f(x,y) = y\sin(xy)$

c- $f(x,y) = \sqrt{x^2+y^2}$

Solution

a- $\frac{\partial z}{\partial x} = 2x + 3y$ and $\frac{\partial z}{\partial y} = 3x + 1$

b- $\frac{\partial z}{\partial x} = y^2\cos(xy)$ and $\frac{\partial z}{\partial y} = xy\cos(xy) + \sin(xy)$

c- $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ and $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

Example: Given $f(x,y) = 3x^2 - 2xy^2 + 2$, find $f_x(-3,0)$ and $f_y(2,-1)$?
Solution
We have $f_x = 6x - 2y^2$ and $f_y = -4xy$ so $f_x(-3,0) = 6(-3) - 2(0)^2 = -18$ and $f_y(2,-1) = -4(2)(-1) = 8$.

1.3.1 Higher Order Derivatives

We can form the second order derivative with respect to x where
$\frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial x}\Big) = \frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial y}\Big) = \frac{\partial^2 z}{\partial x \partial y}$.
Similarly, we can form second order derivative with respect to y where
$\frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial y}\Big) = \frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial x}\Big) = \frac{\partial^2 z}{\partial y \partial x}$.
Note that in general, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.
$\frac{\partial^2 z}{\partial x^2} = z_{xx}$ and $\frac{\partial^2 z}{\partial x \partial y} = z_{xy}$
Similarly, $\frac{\partial^2 z}{\partial y^2} = z_{yy}$ and $\frac{\partial^2 z}{\partial y \partial x} = z_{yx}$
Example: Verify that $w_{yx} = w_{xy}$ if $w(x, y) = x^2 - xy + y^2$?
Solution
We have $w_x = 2x - y$ and $w_{xy} = -1$. We have also $w_y = -x + 2y$ and $w_{yx} = -1$. Thus, $w_{yx} = -1 = w_{xy}$
Example: Find $\frac{\partial^4 f}{\partial s \partial r \partial s \partial t}$ if $f(r, s, t) = 1 - 2rs^2t + r^2s$?
Solution

We first differentiate with respect to the variable s, then r, then s again , and finally with respect to t. We have

$$\begin{aligned}
\frac{\partial f}{\partial s} &= -4rst + r^2, \\
\frac{\partial^2 f}{\partial s \partial r} &= -4st + 2r, \\
\frac{\partial^3 f}{\partial s \partial r \partial s} &= -4t, \\
\frac{\partial^4 f}{\partial s \partial r \partial s \partial t} &= -4.
\end{aligned}$$

Example: If $z = e^{x^2+y^2}$, then show that $yz_x - xz_y = 0$?
Solution
We first need to find z_x and z_y. So, $z_x = 2xe^{x^2+y^2}$ and $z_y = 2ye^{x^2+y^2}$. Thus, by substituting into $yz_x - xz_y$ we get

$$\begin{aligned}
yz_x - xz_y &= 2yxe^{x^2+y^2} - 2yxe^{x^2+y^2}, \\
&= 0.
\end{aligned}$$

Example: If $z = f(x + cy) + g(x - cy)$, then show that $c^2 z_{xx} - z_{yy} = 0$?
Solution

Assume $u = x + cy$ and $v = x - cy$ so we have $u_x = 1, v_x = 1$ and $u_y = c, v_y = -c$. To find z_{xx}

$$\begin{aligned} z_x &= f^{'}(u)u_x + g^{'}(v)v_x \\ &= f^{'}(x+cy) + g^{'}(x-cy). \end{aligned}$$

Also,

$$\begin{aligned} z_{xx} &= f^{''}(u)u_x + g^{''}(v)v_x \\ &= f^{''}(x+cy) + g^{''}(x-cy). \end{aligned}$$

Now, to find z_{yy}

$$\begin{aligned} z_y &= f^{'}(u)u_y + g^{'}(v)v_y \\ &= cf^{'}(x+cy) - cg^{'}(x-cy). \end{aligned}$$

Also,

$$\begin{aligned} z_{yy} &= cf^{''}(u)u_x - cg^{''}(v)v_x \\ &= c^2f^{''}(x+cy) + c^2g^{''}(x-cy). \end{aligned}$$

Thus, by substituting into $c^2z_{xx} - z_{yy}$ we get

$$\begin{aligned} c^2z_{xx} - z_{yy} &= c^2f^{''}(x+cy) + c^2g^{''}(x-cy) - c^2f^{''}(x+cy) - c^2g^{''}(x-cy). \\ &= 0. \end{aligned}$$

Example: Consider a function $T = \ln(\sqrt{r^2+s^2})$. Prove that $r\dfrac{\partial T}{\partial r} + s\dfrac{\partial T}{\partial s} = 1$.
Solution

We know that $T = \dfrac{1}{2}\ln(r^2+s^2)$ then

$$\begin{aligned} \frac{\partial T}{\partial r} &= \frac{1}{2}\Big[\frac{2r}{r^2+s^2}\Big] \\ &= \frac{r}{r^2+s^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial T}{\partial s} &= \frac{1}{2}\Big[\frac{2s}{r^2+s^2}\Big] \\ &= \frac{s}{r^2+s^2}. \end{aligned}$$

Thus, by substituting into $r\dfrac{\partial T}{\partial r} + s\dfrac{\partial T}{\partial s}$

$$r\Big[\frac{r}{r^2+s^2}\Big] + s\Big[\frac{s}{r^2+s^2}\Big] = \frac{r^2+s^2}{r^2+s^2} = 1.$$

1.4 Maxima and Minima

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous at (a, b) (critical point) and $f_x(a, b) = f_y(a, b) = 0$, then

(i) $f(x, y)$ has a local maximum at (a, b) if

$$f_{xx}(a, b) < 0 \text{ and } f_{xx}(a, b)f_{yy}(a, b) - \Big[f_{xy}(a, b)\Big]^2 > 0.$$

(ii) $f(x, y)$ has a local minimum at (a, b) if

$$f_{xx}(a, b) > 0 \text{ and } f_{xx}(a, b)f_{yy}(a, b) - \Big[f_{xy}(a, b)\Big]^2 > 0.$$

(iii) $f(x, y)$ has a saddle point at (a, b) if

$$f_{xx}(a, b)f_{yy}(a, b) - \Big[f_{xy}(a, b)\Big]^2 < 0.$$

Note that to find the critical points of $f(x, y)$, we suppose both $f_x(x, y)$ and $f_y(x, y) = 0$, then we solve the equation to x and y.
Example: Find local maxima, local minima and saddle points of the functions

(i) $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$

(ii) $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

(iii) $f(x, y) = x^2 + xy + y^2 + 2y + 5$

Solution
(i) We need firstly to find the critical points of $f(x, y)$, where

$$f_x(x, y) = 2x + y + 3 = 0 \text{ and } f_y(x, y) = x + 2y - 3 = 0.$$

Thus, there is one critical point $(-3, 3)$. Also, we have

$$f_{xx}(-3, 3) = 2, f_{yy}(-3, 3) = 2, \text{ and } f_{xy}(-3, 3) = 1$$

so $f_{xx}(-3, 3) = 2 > 0$ and

$$f_{xx}(-3, 3)f_{yy}(-3, 3) - \Big[f_{xy}(-3, 3)\Big]^2 > 0$$
$$(2)(2) - (1)^2 = 3 > 0.$$

Thus, $f(x, y)$ has a local minimum at $(-3, 3)$ which is $f(-3, 3) = -5$.

(ii) We need firstly to find the critical points of $f(x, y)$, where

$$f_x(x, y) = y - 2x - 2 = 0 \text{ and } f_y(x, y) = x - 2y - 2 = 0.$$

Thus, there is one critical point $(-2, -2)$. Also, we have

$$f_{xx}(-2, -2) = -2, f_{yy}(-2, -2) = -2, \text{ and } f_{xy}(-2, -2) = 1$$

so $f_{xx}(-2, -2) = -2 < 0$ and

$$f_{xx}(-2, -2)f_{yy}(-2, -2) - \Big[f_{xy}(-2, -2)\Big]^2 > 0$$
$$(-2)(-2) - (1)^2 = 3 > 0.$$

Thus, $f(x, y)$ has a local maximum at $(-2, -2)$ which is $f(-2, -2) = 8$.

(iii) We need firstly to find the critical points of $f(x, y)$, where

$$f_x(x, y) = 2x + y + 3 = 0 \text{ and } f_y(x, y) = x + 2 = 0.$$

Thus, there is one critical point $(-2, 1)$. Also, we have

$$f_{xx}(-2, 1) = 2, f_{yy}(-2, 1) = 0, \text{ and } f_{xy}(-2, 1) = 1$$

so

$$f_{xx}(-2, 1)f_{yy}(-2, 1) - \Big[f_{xy}(-2, 1)\Big]^2 < 0$$
$$(2)(0) - (1)^2 = -1 < 0.$$

Thus, $f(x, y)$ has a saddle point at $(-2, 1)$.

1.5 Exercises

Q_1 : **Find the limits of the functions below**

$$\lim_{(x,y)\longrightarrow(0,4)} \frac{x}{\sqrt{y}}$$

$$\lim_{(x,y)\longrightarrow(1,1)} \frac{x^2-y^2}{x-y}$$

$$\lim_{(x,y)\longrightarrow(2,2)} \frac{x+y-4}{\sqrt{x+y}-2}$$

$$\lim_{(x,y)\longrightarrow(3,4)} \frac{\sqrt{x}-\sqrt{y-1}}{x-y-1}$$

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{\sqrt{x^2-2y^3-2xy^2+xy}}{\sqrt{x+y}}$$

Q_2 : **Find** $\frac{\partial z}{\partial x}$ **and** $\frac{\partial z}{\partial y}$

(i) $z = 5xy - 7x^2 - y^2 + 3x - 6y$

(ii) $z = \frac{x}{x^2+y^2}$

(iii) $z = e^{xy}\ln(y)$

(v) $z = tan^{-1}(\frac{y}{x})$

Q_3 : **Verify that** $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$ **if**

(i) $F = x\sin(y) + y\sin(x) + xy$

(ii) $F = \ln(2x+3y)$

(iii) $F = e^x + x\ln(y) + y\ln(x)$

Q_4 : **Prove that**

(i) If $G(r,\Theta) = \sqrt{r^2+\Theta^2}$, then $rG_r + \Theta G_\Theta = G$.

(ii) If $W = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then $xW_x + yW_y + zW_z = 0$.

(iii) If $V = f(s+t) + sg(s+t)$, then $V_{ss} - V_{st} + V_{tt} = 0$.

Q_5 : **Find local maxima, local minima and saddle points of the functions below**

(1) $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

(2) $f(x, y) = 3y^2 - 3x^2 - 2y^3 - 3x^2 + 6xy$

(3) $f(x, y) = x^3 + 3x^2 + y^3 - 3y^2 - 8$

Chapter 2

Double Integrals

In this chapter, we will learn how to evaluate double integrals. Let $z = f(x, y)$ be a function which is continuous on closed region $D : a_1(y) \leq x \leq a_2(y), b_1(x) \leq y \leq b_2(x)$. We may interpret the double integral of z over D as the volume. So, we define this volume to be

$$\text{volume} = \int\int_D f(x,y)dA = \int_{y=b_1(x)}^{y=b_2(x)} \int_{x=a_1(y)}^{x=a_2(y)} f(x,y)dxdy.$$

Or

$$\text{volume} = \int\int_D f(x,y)dA = \int_{x=a_1(y)}^{x=a_2(y)} \int_{y=b_1(x)}^{y=b_2(x)} f(x,y)dydx.$$

2.1 Properties of Double integrals

If $f(x, y)$ and $g(x, y)$ are continuous on the region D then

$$\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA \text{ for any number } c.$$

$$\int\int_D \Big[f(x,y) \mp g(x,y)\Big]dA = \int\int_D f(x,y)dA \mp \int\int_D g(x,y)dA.$$

$$\int\int_D f(x,y)dA = \int\int_{D_1} f(x,y)dA + \int\int_{D_2} f(x,y)dA,$$

if D is the union of two regions $D = D_1 \cup D_2$.
Note that if $f(x, y)$ is continuous over the region $D : a \leq x \leq b, c \leq y \leq d$, then

$$\int\int_D f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

Example: Calculate

$$\int\int_D f(x,y)dA \text{ for } f(x,y) = 1 - 6x^2y \text{ and } D: 0 \le x \le 2, -1 \le y \le 1.$$

Solution

$$\begin{aligned}\int\int_D f(x,y)dA &= \int_{-1}^{1}\int_{0}^{2}(1-6x^2y)dxdy\\ &= \int_{-1}^{1}\Big[x - 2x^3y\Big]_0^2 dy,\\ &= \int_{-1}^{1}(2-16y)dy,\\ &= \Big[2y - 8y^2\Big]_{-1}^{1}\\ &= 4.\end{aligned}$$

Now by changing the order of integration gives the same answer:

$$\begin{aligned}\int_0^2\int_{-1}^{2}(1-6x^2y)dydx &= \int_0^2\Big[y - 3x^2y^2\Big]_{-1}^{1}dx,\\ &= \int_0^2[(1-3x^2)-(-1-3x^2)]dx\\ &= 4.\end{aligned}$$

Example: Find the volume of solid bounded by the lines $x = 4$ and $y = 8$ and the surface $z = 4 - \dfrac{x}{2} + \dfrac{y^2}{16}$.

Solution

By drawing (3-dimensional) the lines $x = 4$ and $y = 8$ plane and the surface $z = 4 - \dfrac{x}{2} + \dfrac{y^2}{16}$ which gives (see Figure 1)

$$\begin{aligned}
\text{volume} &= \int\int_D z dA \\
&= \int_0^4 \int_0^8 (4 - \frac{x}{2} + \frac{y^2}{16}) dy dx \\
&= \int_0^4 \Big[4y - \frac{yx}{2} + \frac{y^3}{48}\Big]_0^8 dx, \\
&= \int_0^2 (\frac{128}{3} - 2x^2) dx \\
&= 416/3.
\end{aligned}$$

Figure 1

Example: Find the volume of solid bounded by the lines $x = 2$ and $y = 1$ and the plane $z = 4 - x - y$.
Solution
By drawing the lines $x = 1$ and $y = 1$ and the plane
$z = 4 - x - y$ which gives (see Figure 2)

$$\begin{aligned}
\text{volume} &= \int\int_D z dA \\
&= \int_0^2 \int_0^1 (4 - x - y) dy dx \\
&= \int_0^1 \Big[4y - xy + \frac{y^2}{2}\Big]_0^2 dx, \\
&= \int_0^2 (\frac{7}{2} - x) dx \\
&= 5.
\end{aligned}$$

Figure 2

2.2 Areas and Centers of Mass

In this section, we show how to use double integrals to calculate the areas of bounded regions in the plane. Also, we study the physical problem of finding the center of mass of a thin flat plate covering a region in plane.

2.2.1 Areas of bounded regions in the plane

We define the area of a closed, bounded plane region D is $A = \int\int_D dA$

Example: Find the area of the region D bounded by $y = x$ and $y = x^2$ in the first quadrant?

Solution

We sketch the region D (see Figure 3) and calculate the area as

$$\begin{aligned}
\text{A} &= \int\int_D dA \\
&= \int_0^1 \int_{x^2}^{x} dydx \\
&= \int_0^1 \Big[y\Big]_{x^2}^{x} dx, \\
&= \int_0^1 (x - x^2)dx \\
&= 1/6.
\end{aligned}$$

Figure 3

Example: Find the area of the region D bounded by $y = x + 2$ and $y = x^2$ in the first quadrant?

Solution

We sketch the region D (see Figure 4) and calculate the area as

$$\begin{aligned}
\text{A} &= \int\int_D dA \\
&= \int_{-1}^2 \int_{x^2}^{x+2} dydx \\
&= \int_{-1}^2 \Big[y\Big]_{x^2}^{x} dx, \\
&= \int_{-1}^2 (x - x^2 + 2)dx \\
&= 9/2.
\end{aligned}$$

Figure 4

Now if we change the order of integration what will we see?

2.2.2 Centers of mass for thin flat plates

We assume the distribution of mass in such a plate to be continuous. A material's density function, denoted by $\delta(x, y)$ is the mass per unit area. The mass of a plate is obtained by integrating the density function over the region D forming the plate . The first moment about an axis is calculated by integrating over D the distance from the axis times the density.

The center of mass formulas for thin flat plates covering a region D in the $xy-$ plane

$$\text{Mass:} \quad M = \int\int_D \delta(x,y)dA, \text{ where } \delta(x,y) \text{ is density function}$$

$$\text{First moments:} \quad M_x = \int\int_D y\delta(x,y)dA, \text{ and } M_y = \int\int_D x\delta(x,y)dA$$

$$\text{Center of mass:} \quad \overline{x} = \frac{M_y}{M} \text{ and } \overline{y} = \frac{M_x}{M}.$$

Example: Find a center of mass of a thin plat of density $\delta(x, y) = 6x+6y+6$ bounded by $x = 1$ and $y = 2x$ in the first quadrant?

Solution

We sketch the region $x = 1$ and $y = 2x$ (see Figure 5) and calculate the Mass as

$$\begin{aligned}
\text{Mass:} \quad M &= \int\int_D \delta(x,y)dA, \text{ where } \delta(x,y) \text{ is density function} \\
&= \int_0^1 \int_0^{2x} (6x + 6y + 6)dydx \\
&= \int_0^1 \Big[6xy + 3y^2 + 6y\Big]_0^{2x} dx, \\
&= \int_0^1 (24x^2 + 12x)dx \\
&= 14.
\end{aligned}$$

Figure 5

The first moments about the $x-$ axis is

$$\begin{aligned}
\text{First moments:} \quad M_x &= \int\int_D y\delta(x,y)dA \\
&= \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y)dydx \\
&= \int_0^1 \Big[3xy^2 + 3y^3 + y^2\Big]_0^{2x} dx, \\
&= \int_0^1 (28x^3 + 12x^2)dx \\
&= 11.
\end{aligned}$$

The first moments about the $y-$ axis is

$$\begin{aligned}
\text{First moments:} \quad M_y &= \int\int_D x\delta(x,y)dA \\
&= \int_0^1 \int_0^{2x} (6x^2 + 6xy + 6x)dydx \\
&= \int_0^1 \Big[6x^2y + 3xy^2 + 6xy\Big]_0^{2x} dx, \\
&= \int_0^1 (24x^3 + 12x^2)dx \\
&= 10.
\end{aligned}$$

Thus, the center of mass are

$$\overline{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7} \text{ and } \overline{y} = \frac{M_x}{M} = \frac{11}{14}.$$

2.3 Double Integral in Polar form

This section shows how to change a Cartesian integral $\int\int_D f(x,y)dA$, into a polar integral $\int\int_G \delta(r,\theta)rdrd\theta$, which is easier to evaluate.
Recall that if P is a point in two-dimensional space. Then the polar coordinates of P are (r,θ) if P is r units from the origin, and the ray from the origin to P makes an angle θ with the positive $x-$ axis (see figure 6). The relationship between polar coordinates and cartesian coordinates are

$$x = r\cos(\theta), y = r\sin(\theta), r = \sqrt{x^2 + y^2} \text{ and } \theta = tan^{-1}(\frac{y}{x})$$

Figure 6

Polar integration formula

$$\int\int_D f(x,y)dA = \int\int_G (r\cos\theta, r\sin\theta)\, rdrd\theta \text{ where } G : \alpha \leq \theta \leq \beta, b(\theta) \leq r \leq a(\theta).$$

Note that for all point (r, θ) in the region G:

$$r \geq 0 \text{ and } 0 \leq \theta \leq 2\Pi.$$

Example: Evaluate $\int\int_D e^{x^2+y^2} dydx$, where D is region bounded by the $x-$ axis and the curve $y = \sqrt{1-x^2}$?
Solution
There is no way to integrate $e^{x^2+y^2}$ with respect to either x or y. Thus, we need to use the polar coordinates which enables us to evaluate the integral as

$$y = \sqrt{1-x^2} \Longrightarrow y^2 = 1-x^2 \Longrightarrow x^2+y^2 = 1 \Longrightarrow r^2 = 1 \Longrightarrow r = \mp 1 \Longrightarrow r = 1$$

We sketch the region $y = \sqrt{1-x^2}$ and $x-$axis (see Figure 7) so

$$\begin{aligned}
\int\int_D e^{x^2+y^2} dydx &= \int_0^{\Pi}\int_0^1 (re^{r^2})drd\theta \\
&= \int_0^{\Pi} \left[\frac{1}{2}e^{r}\right]_0^1 d\theta, \\
&= \frac{1}{2}\int_0^1 (e-1)d\theta \\
&= \frac{\Pi}{2}(e-1).
\end{aligned}$$

Figure 7

Example: Evaluate $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}} dydx.$
Solution
There is no way to integrate $\frac{2}{1+\sqrt{x^2+y^2}}$ with respect to either x or y. By using a polar coordinates gives

$$y = -\sqrt{1-x^2} \Longrightarrow y^2 = 1-x^2 \Longrightarrow x^2+y^2 = 1 \Longrightarrow r^2 = 1 \Longrightarrow r = \mp 1 \Longrightarrow r = 1$$

We sketch $y = -\sqrt{1-x^2}, y = 0, x = 0$ and $x = -1$ (see Figure 8) so

$$\begin{aligned}
\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}} &= \int_{\Pi}^{3\frac{\Pi}{2}}\int_{0}^{1} \frac{2r}{1+r} dr d\theta \\
&= \int_{\Pi}^{3\frac{\Pi}{2}} \Big[r - \ln(r+1) \Big]_{0}^{1} d\theta, \\
&= \frac{1}{2}\int_{0}^{1} (1-\ln 2) d\theta \\
&= (1-\ln 2)\Pi.
\end{aligned}$$

Figure 8

Example: Evaluate by using polar integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2) dy dx.$

Solution

2.4 Exercises

Q_1 : **Evaluate each of the following double integral**

(1) $\int\int_D f(x,y)dA$ for $f(x,y) = x^2y - 2xy$ and $D : 0 \leq x \leq 3, -2 \leq y \leq 0.$

(2) $\int\int_D f(x,y)dA$ for $f(x,y) = x \sin y$ and $D : 0 \leq x \leq \Pi, 0 \leq y \leq x.$

(3) $\int\int_D f(x,y)dA$ for $f(x,y) = e^{x+y}$ and $D : 0 \leq x \leq \ln y, 1 \leq y \leq \ln 8.$

Q_2 : **Find the volume of the solid cut from the first actant by the surface** $z = 4 - x^2 - y$**?**

Q_3 : **Find the area of the region** D **for each of following by using the double integral**

(1) D : The lines $x + y = 2, x = 0$ and $y = 0.$

(2) D : The parabola $x = -y^2$ and the line $y = x + 2.$

(3) D : The curve $y = e^x$ and the lines $y = 0, x = 0$ and $x = \ln 2.$

Q_4 : **Evaluate**

(1) $\int_0^1 \int_y^1 \frac{\sin x}{x} dxdy$

(2) $\int_0^1 \int_x^1 e^{y^2} dydx$

(3) $\int_0^3 \int_{x^2}^9 x \cos y^2 dydx$

(4) $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} dxdy$

Q_5 : **Find a center of mass of a thin plat of density** $\delta(x,y) = 3$ **bounded by** $y = 2 - x^2, y = x$ **and** $y-$ **axis?**

Q_6 : **Evaluate by using polar integral** $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx.$

Chapter 3

Introduction to Vectors

In weather reports, we hear that the wind has speed and direction. One way of representing a wind of $30k$ (kilometers) per hour from the southwest, is to draw arrow aimed in the direction in which the wind blow, scaled.

Definition: A vector is a direction line segment. The direction line segment $V = \overrightarrow{PQ}$ has initial point P and terminal point Q.

If V is two-dimensional with $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ as points in the plane then

$$V =< x_2 - x_1, y_2 - y_1 > .$$

Similarly, if V is three-dimensional with $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ as points in the plane then

$$V =< x_2 - x_1, y_2 - y_1, z_2 - z_1 > .$$

Definition: The length of the vector (three-dimensional) $V = \overrightarrow{PQ}$ is the non-negative number

$$| V |= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example: If $P(1, 0, 1), Q(2, 0, 3)$ and $R(3, 5, 6)$ are points in three-dimensional, then plot the points and write down the vectors to each of the following $\overrightarrow{PQ}, \overrightarrow{PR}$ and $\overrightarrow{RQ}$?

Solution

For plot see Figure 9 and for find $\overrightarrow{PQ}, \overrightarrow{PR}$ and $\overrightarrow{RQ}$ we have

$$
\begin{aligned}
V_1 = \overrightarrow{PQ} &=< x_2 - x_1, y_2 - y_1, z_2 - z_1 > \\
&=< 2-1, 0-0, 3-1 > \\
&=< 1, 0, 2 > . \\
V_2 = \overrightarrow{PR} &=< x_2 - x_1, y_2 - y_1, z_2 - z_1 > \\
&=< 3-1, 5-0, 6-1 > \\
&=< 2, 5, 5 > . \\
V_3 = \overrightarrow{RQ} &=< x_2 - x_1, y_2 - y_1, z_2 - z_1 > \\
&=< 2-3, 0-5, 3-6 > \\
&=< 2, 5, 5 > .
\end{aligned}
$$

Figure 9

Example: Find length of the vectors $V_1 = \overrightarrow{PQ}, V_2 = \overrightarrow{PR}$ and $V_3 = \overrightarrow{RQ}$ where $P(1,0,1), Q(2,0,3)$ and $R(3,5,6)$?

Solution

$$
\begin{aligned}
\mid V_1 \mid = \overrightarrow{PQ} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \sqrt{(2-1)^2 + (0-0)^2 + (3-1)^2} \\
&= \sqrt{5}. \\
\mid V_2 \mid = \overrightarrow{PR} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \sqrt{(3-1)^2 + (5-0)^2 + (6-1)^2} \\
&= \sqrt{54} \\
&= 3\sqrt{6}. \\
\mid V_3 \mid = \overrightarrow{RQ} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \sqrt{(2-3)^2 + (0-5)^2 + (3-6)^2} \\
&= \sqrt{35}.
\end{aligned}
$$

3.0.1 Vector Algebra Operations

Two principal operations involving vectors are vector addition and scalar multiplication. A scalar is simple a real number.
Definition: Let $V_1 =< x_1, x_2, \ldots, x_n >$ and $V_2 =< y_1, y_2, \ldots, y_n >$ be

vectors with $k \in \mathbb{R}$. Then

$$V_1 \mp V_2 =< x_1 \mp y_1, x_2 \mp y_2, \ldots, x_n \mp y_n >,$$

and

$$kV_1 =< kx_1, kx_2, \ldots, kx_n > .$$

Note that the zero vector is written $\overrightarrow{0} =< 0, 0, \ldots, 0 >$.

Properties of Vector Operations

Let V_1, V_2 and V_3 be vectors and $k \in \mathbb{R}$. Then

(1) $V_1 + V_2 = V_2 + V_1$

(2) $V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$

(3) $V_1 + \overrightarrow{0} = V_1$

(4) $V_1 - V_1 = \overrightarrow{0}$

(5) $k(V_1 + V_2) = kV_1 + kV_2$

Unit Vectors:

Let $V = (x_1, x_2, \ldots, x_n)$ be a non-zero vector. Then, a vector of length 1 is called a unit vector and the unit vector is denoted by

$$\text{Unit Vector of } V = \left\langle \frac{x_1}{| V |}, \frac{x_2}{| V |}, \ldots, \frac{x_n}{| V |} \right\rangle.$$

The three most important unit vectors are

$$i =< 1, 0, 0 >, j =< 0, 1, 0 >, \text{ and } k =< 0, 0, 1 > .$$

Thus, any vector $V =< x_1, x_2, \ldots, x_n >$ can be written as linear of the standard unit vectors as follows

$$\begin{aligned} V =< x_1, x_2, x_3 > &=< x_1, 0, 0 > + < 0, x_2, 0 > + < 0, 0, x_3 > \\ &= x_1 < 1, 0, 0 > + x_2 < 0, 1, 0 > + x_3 < 0, 0, 1 > \\ &= x_1 i + x_2 j + x_3 k. \end{aligned}$$

Dot Product:

The dot product of two vectors is defined as following:

If $V_1 =< x_1, x_2, \ldots, x_n >$ and $V_2 =< y_1, y_2, \ldots, y_n >$ then $V_1 \cdot V_2 = x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_n \cdot y_n$

Properties of Dot Product

Let V_1, V_2 and V_3 be vectors and $k \in \mathbb{R}$. Then

(1) $V_1 \cdot V_2 = V_2 \cdot V_1$

(2) $V_1 \cdot (V_2 + V_3) = (V_1 \cdot V_2) + (V_1 \cdot V_3)$

(3) $V_1 \cdot \overrightarrow{0} = \overrightarrow{0}$

(4) $V_1 \cdot V_1 = |V_1|^2$

(5) $k(V_1 \cdot V_2) = (kV_1) \cdot V_2 = V_1 \cdot (kV_2)$

Angle Between Vectors

We define the angle between two non-zero vectors V and U as

$$\cos\theta = \frac{V \cdot U}{|V||U|}$$

Note that $0 < \theta < \Pi$. If they point in same direction, then the angle between them is 0 , and Π if they point in opposite direction.

Example: If $V = i - j - k$ and $U = 2i + j + k$, then find

1)The unit vector of V and U.
2)The angle between two vectors V and U.
Solution:
1) We will find the unit vector of V and U.

$$|V| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{3} \text{ and } |U| = \sqrt{(2)^2 + (1)^2 + (1)^2} = \sqrt{6}$$

So, the unit vector of V is $V = \frac{i}{\sqrt{3}} - \frac{j}{\sqrt{3}} - \frac{k}{\sqrt{3}}$ and $U = \frac{2i}{\sqrt{6}} + \frac{j}{\sqrt{6}} + \frac{k}{\sqrt{6}}$.

2)We will find the angle between two vectors V and U.

$$\begin{aligned} V \cdot U &= (1)(2) + (-1)(1) + (-1)(1) = 2 - 1 - 1 = 0 \\ |V| &= \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{3} \\ |U| &= \sqrt{(2)^2 + (1)^2 + (1)^2} = \sqrt{6} \\ \cos\theta &= \frac{V \cdot U}{|V||U|} = \frac{0}{\sqrt{3}\sqrt{6}} \end{aligned}$$

So, $\theta = \cos^{-1}(0) = \frac{\Pi}{2}$.
Note that if the dot product of two vectors is zero, then they are perpendicular

$$V \cdot U = 0 \Longleftrightarrow V \perp U.$$

Example :Find the angle between two vectors $A = i + \sqrt{3}j$ and $B = (1 - \sqrt{3})i + (1 + \sqrt{3})j$?

Solution:

$$\begin{aligned} A \cdot B &= (1)(1 - \sqrt{3}) + (\sqrt{3})(1 + \sqrt{3}) = 4 \\ |A| &= \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2 \\ |B| &= \sqrt{(1 - \sqrt{3})^2 + (1 + \sqrt{3})^2} = \sqrt{8} = 2\sqrt{2} \\ \cos\theta &= \frac{A \cdot B}{|A||B|} = \frac{4}{4\sqrt{2}} \end{aligned}$$

So, $\theta = \cos^{-1}(\frac{1}{\sqrt{2}}) = \frac{\Pi}{4}$.
Example: Find a if $(V = ai + 5j + 6k) \perp (U = ai + aj + k)$?
Solution:
We have $V \cdot U = 0 \Longleftrightarrow V \perp U$. So,

$$\begin{aligned} V \cdot U &= (a)(a) + (5)(a) + (6)(1) \\ &= a^2 + 5a + 6 \\ &= (a + 3)(a + 2) \end{aligned}$$

That is $(a + 3)(a + 2) = 0 \Longrightarrow a = -3$ or $a = -2$.

The Cross Product The cross product of two vectors is defined as follows. If $V_1 = x_1 i + x_2 j + x_3 k$ and $V_2 = y_1 i + y_2 j + y_3 k$, then we have the vector

$$V_1 \times V_2 \quad = \quad \begin{array}{ccc} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array}$$

Properties of the Cross Product

Let V_1, V_2 and V_3 be vectors and $s, r \in \mathbb{R}$. Then

(1) $V_1 \times V_2 = -(V_2 \times V_1)$

(2) $V_1 \times (V_2 + V_3) = (V_1 \times V_2) + (V_1 \times V_3)$

(3) $(V_2 + V_3) \times V_1 = (V_2 \times V_2) + (V_3 \times V_1)$

(4) $\overrightarrow{0} \times V_1 = \overrightarrow{0}$

(5) $(sV_1) \times (rV_2) = (sr)(V_1 \times V_2)$

Example: Find $V \times U$ if $V = 4i + 2j - k$ and $U = -3i + 2j + 3k$?
Solution:

$$V \times U \quad = \quad \begin{array}{ccc} i & j & k \\ 4 & 2 & -1 \\ -3 & 2 & 3 \end{array} \quad = 8i - 9j + 14k$$

Note that non-zero vectors V_1 and V_2 are parallel if and only if $V_1 \times V_2 = 0$
Example: Show that the vector $V = 5i - j + k$ is parallel to the vector $U = -15i + 3j - 3k$?
Solution:

$$V \times U \quad = \quad \begin{array}{ccc} i & j & k \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{array} \quad = 0.$$

Note that if

$$V_1 = x_1 i + x_2 j + x_3 k,\ V_2 = y_1 i + y_2 j + y_3 k,\ \text{and } V_3 = z_1 i + z_2 j + z_3 k,$$

then

$$(V_1 \times V_2) \cdot V_2 \quad = \quad \begin{array}{ccc} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{array} \quad = \text{constant}$$

Example: Verify that $(U \times V) \cdot W = (V \times W) \cdot U$ where

$$U = i - j + k, V = 2i + j - 2k \text{ and } W = -i + 2j - k$$

Solution:

$$(U \times V) \cdot W \quad = \begin{matrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{matrix} \quad = 4$$

and

$$(V \times W) \cdot U \quad = \begin{matrix} 2 & 1 & -2 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{matrix} \quad = 4.$$

Vector Field

A vector field on a domain in space is a function that assigns a vector to each point in the domain. A field of three dimensional vector might have a formula like

$$f(x, y, z) = M(x, y, z)i + N(x, y, z)j + P(x, y, z)k.$$

Gradient Field

The gradient field of a differentiable $f(x, y, z)$ is the field of gradient vectors

$$F = \nabla f = f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k$$

Divergence

The divergence of a vector filed $f = M(x, y, z)i + N(x, y, z)j + P(x, y, z)k$ is

$$div f = f \cdot \nabla = M_x(x, y, z) + N_y(x, y, z) + P_z(x, y, z)$$

Example: Find the gradient field and the divergence of $f(x, y, z) = xyz$.
Solution:
The gradient field of f is

$$\nabla f = f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k = yzi + xzj + xyk$$

and the divergence is

$$div f = f \cdot \nabla = M_x(x, y, z) + N_y(x, y, z) + P_z(x, y, z) = 0$$

Chapter 4

Differential Equations

Basically, a differential equation is a relationship between a function and its derivatives. We will consider first-ordinary differential equation (ode) as

$$(1) y + x\frac{dy}{dx}, \quad (2) \frac{dy}{dx} = \sin x \sec y, \quad (3) \frac{dx}{dy} = \frac{x^2 + xy}{y^2}.$$

4.1 Solutions of a Differential Equation

A solution of differential equation is any function which satisfies the differential equation identically.

Example: Consider the differential equation

$$\frac{d^2y}{dx^2} - 4y = e^x,$$

verify that $y = -\frac{1}{3}e^x$ is a solution.

Solution:
Note that

$$\frac{dy}{dx} = -\frac{1}{3}e^x \text{ and } \frac{d^2y}{dx^2} = -\frac{1}{3}e^x.$$

Now, by substituting into the differential equation

$$\begin{aligned}\frac{d^2y}{dx^2} - 4y =& -\frac{1}{3}e^x - 4(-\frac{1}{3}e^x) \\ =& e^x(\frac{-1}{3} + \frac{4}{3}) \\ =& e^x.\end{aligned}$$

4.2 Method of solution for first-ordinary differential equation

Method1: Variables Separable Equation

A differential equation which can be written in the form

$$\frac{dy}{dx} = f(x)g(x).$$

We solve a variables separable equation by separating the variables and integrating.

Example: Solve the following equations

$$(1)\frac{dy}{dx} = \frac{x - xy^2}{x^2y - y} \quad (2)xydy + (x^2 + 1)dx = 0.$$

Solution:

solution for (1)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x - xy^2}{x^2y - y} \\
\frac{dy}{dx} &= \frac{x(1 - y^2)}{y(x^2 - 1)} \\
\frac{ydy}{1 - y^2} &= \frac{xdx}{x^2 - 1} \\
-\frac{1}{2}\ln|1 - y^2| &= \frac{1}{2}\ln|x^2 - 1| + \ln c \\
\ln|1 - y^2| &= -\ln|x^2 - 1| - 2\ln c \\
\ln|1 - y^2| &= -\ln|x^2 - 1| + A, \text{ by putting } -2\ln c = A \\
1 - y^2 &= \frac{1}{A(x^2 - 1)}.
\end{aligned}$$

Now solution for (2),

$$\begin{aligned}
xydy + (x^2 + 1)dx = 0 &\Longrightarrow ydy + \frac{(x^2 + 1)}{x}dx = 0 \\
&\Longrightarrow ydy = -xdx - \frac{dx}{x} \\
&\Longrightarrow \frac{y^2}{2} = -\frac{x^2}{2} - \ln x + c \\
&\Longrightarrow y^2 = x^2 - 2\ln x + A, \text{ where } A = 2c.
\end{aligned}$$

Method2: Homogeneous Equation

If we can write the equation in the form

$$\frac{dy}{dx} = f(\frac{y}{x}) \text{ or } \frac{dx}{dy} = f(\frac{x}{y})$$

then we say the differential equation is homogeneous. For homogeneous equation can be transformed to a variables separable equation by putting $v = \frac{y}{x}$ or $v = \frac{x}{y}$. Then,

$$v = \frac{y}{x} \Longrightarrow y = xv \Longrightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

or

$$v = \frac{x}{y} \Longrightarrow y = xv \Longrightarrow \frac{dx}{dy} = y\frac{dv}{dx} + v.$$

Example: Solve the following equations

$$(1)\frac{dy}{dx} = \frac{x-y}{x+y} \quad (2)\frac{dx}{dy} = \frac{y^2 + y^2 e^{(\frac{x}{y})^2} + 2x^2 e^{(\frac{x}{y})^2}}{2xye^{(\frac{x}{y})^2}}.$$

Solution:

(1)let $v = \frac{y}{x}$ that is $y = xv \Longrightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$ so we will have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x-y}{x+y} \\
\frac{dy}{dx} &= \frac{1-y/x}{1+y/x} \\
x\frac{dv}{dx} + v &= \frac{1-v}{1+v}, \text{ by using } v = \frac{y}{x} \text{ and } \frac{dy}{dx} = x\frac{dv}{dx} + v \\
x\frac{dv}{dx} &= \frac{1-v}{1+v} - v \\
x\frac{dv}{dx} &= \frac{1-2v-v^2}{1+v}
\end{aligned}$$

Now, separate variables and solve

$$\begin{aligned}
\frac{(1+v)dv}{1-2v-v^2} &= \frac{dx}{x} \\
-\frac{1}{2}\ln|1-2v-v^2| &= \ln x + \ln c \\
\ln|1-2v-v^2| &= -2\ln x - 2\ln c \\
\ln|1-2v-v^2| + \ln x^2 &= A, \text{ where } A = -2\ln c \\
\ln|x^2(1-2v-v^2)| &= A \\
x^2(1-2(y/x)-(y/x)^2) &= e^A.
\end{aligned}$$

(2)let $v = \frac{x}{y}$ that is $x = yv \Longrightarrow \frac{dx}{dy} = y\frac{dv}{dy} + v$ so we will have

$$\frac{dx}{dy} = \frac{y^2 + y^2 e^{(\frac{x}{y})^2} + 2x^2 e^{(\frac{x}{y})^2}}{2xye^{(\frac{x}{y})^2}}$$
$$\frac{dx}{dy} = \frac{y^2/y^2 + y^2/y^2 e^{(\frac{x}{y})^2} + 2x^2/y^2 e^{(\frac{x}{y})^2}}{2xy/y^2 e^{(\frac{x}{y})^2}}$$

$$y\frac{dv}{dy} + v = \frac{1 + e^{(v)^2} + 2v^2 e^{(v)^2}}{2ve^{(v)^2}}$$
$$y\frac{dv}{dy} = \frac{1 + e^{(v)^2} + 2v^2 e^{(v)^2}}{2ve^{(v)^2}} - v$$
$$y\frac{dv}{dy} = \frac{1 + e^{(v)^2} + 2v^2 e^{(v)^2} - 2v^2 e^{(v)^2}}{2ve^{(v)^2}}$$
$$y\frac{dv}{dy} = \frac{1 + e^{(v)^2}}{2ve^{(v)^2}}$$

Now, separate variables and solve

$$\frac{dy}{y} = \frac{2ve^{(v)^2} dv}{1 + e^{(v)^2}}$$
$$\ln|y| = \ln(1 + e^{(v)^2}) + \ln c$$
$$y = (1 + e^{(v)^2})c$$
$$y = (1 + e^{(\frac{x}{y})^2})c.$$

Note that if $y(0) = 1$, then we have

$$y = (1 + e^{(\frac{x}{y})^2})c$$
$$1 = (1 + e^{(\frac{0}{1})^2})c$$
$$1 = (1 + 1)c, \text{ where } e^0 = 1$$
$$c = 1/2.$$

Thus, $y = \frac{1}{2}(1 + e^{(\frac{x}{y})^2})$ if $y(0) = 1$.

Method3: Exact Differential Equation
A differential equation

$$M(x,y)dx + N(x,y)dy = 0,$$

is exact if and only if $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$.

Example: Solve $2xydx + (1+x^2)dy = 0$
Solution:
We have $\underbrace{2xy}_{M(x,y)}\ dx + \underbrace{(1+x^2)}_{N(x,y)}dy = 0$ and here

$$M(x,y) = 2xy \text{ and } N(x,y) = x^2,$$

therefore,

$$\frac{\partial M(x,y)}{\partial y} = 2x = \frac{\partial N(x,y)}{\partial x}.$$

So, this equation is exact. Now,

$$\begin{aligned}\int_x M(x,y)dx =& \int 2xydx \\ g(x,y) =& x^2y + A(y)\end{aligned}$$

By differentiating $g(x,y)$ with respect to y we obtain

$$\begin{aligned}g(x,y) =& x^2y + A(y) \\ \frac{\partial g}{\partial y} =& x^2 + A'(y)\end{aligned}$$

Now, $N(x,y) = 1 + x^2 = x^2 + A'(y) \Longrightarrow A'(y) = 1 \Longrightarrow A(y) = y + c$.
The solution to the differential equation, which is given above is

$$g(x,y) = x^2y + y + c.$$

Note that if $g(x,y) = k$ then $A = x^2y + y \Longrightarrow y = \frac{A}{x^2+1}$, where $k - c = A$.

Example: Solve $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

Solution:

We have $\underbrace{(y^2e^{xy^2}+4x^3)}_{M(x,y)}dx+\underbrace{(2xye^{xy^2}-3y^2)}_{N(x,y)}dy=0$ and here

$$M(x,y)=y^2e^{xy^2}+4x^3 \text{ and } N(x,y)=2xye^{xy^2}-3y^2,$$

therefore,

$$\frac{\partial M(x,y)}{\partial y}=2ye^{xy^2}+2xy^3e^{xy^2}=\frac{\partial N(x,y)}{\partial x}.$$

So, this equation is exact. Now,

$$\begin{aligned}\int_x M(x,y)dx=&\int(y^2e^{xy^2}+4x^3)dx\\ g(x,y)=&e^{xy^2}+x^4+A(y)\end{aligned}$$

By differentiating $g(x,y)$ with respect to y we obtain

$$\begin{aligned}g(x,y)=&e^{xy^2}+x^4+A(y)\\ \frac{\partial g}{\partial y}=&2xye^{xy^2}+A'(y)\end{aligned}$$

Now, $N(x,y)=2xye^{xy^2}-3y^2=2xye^{xy^2}+A'(y)\Longrightarrow A'(y)=-3y^2\Longrightarrow A(y)=-y^3+c.$

Thus, $g(x,y)=e^{xy^2}+x^4-y^3+c\Longrightarrow e^{xy^2}+x^4-y^3=A.$

Using Chain Rule with Partial Derivatives

The Chain Rule for functions of a single variable studied in first stage says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and $\frac{dw}{dt}$ can be calculated by the formula $\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$.

For functions of two or more variables the Chain Rule has several forms. The form depends on how many variables are involved, but there are three general cases as a following:

Case1: If $w = f(x, y, z)$ is a differentiable function and $x = x(t), y = y(t), and\ z = z(\mathrm{t})$, are differentiable functions of t, then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example: Prove that $\frac{dw}{dt} = \cos 2t$, if $w = xy, and\ x = \cos t\,, y = \sin t$.

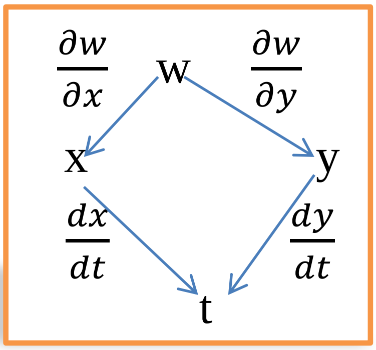
Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

Note that $\frac{\partial w}{\partial x} = y\,, \frac{dx}{dt} = -\sin t\,, \frac{\partial w}{\partial y} = x, and\ \frac{dy}{dt} = \cos t$.

Now, $\frac{dw}{dt} = -y\sin t + x\cos t,$

$$\frac{dw}{dt} = -\sin t \sin t + \cos t \cos t,$$

$$\frac{dw}{dt} = cos^2 t - sin^2 t = \cos 2t.$$



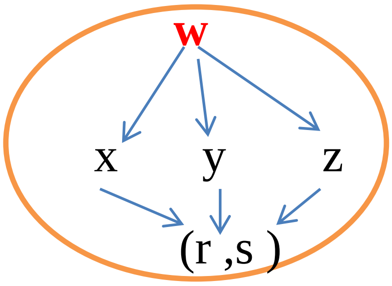
Case2: If $w = f(x, y, z)$ where $x = g(r, s), y = h(r, s), and\ z = k(r, s)$. If all four functions are differentiable functions , then w have partial derivatives with respect to r and s given by the formulas:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Example: Find $\frac{\partial w}{\partial r}$ *and* $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2 + z^2$, *and* $x = r^2 + s,\ y = r\ln s\ ,z = \tan^{-1}(s^2 + r^2)$.

Solution: By using the formulas in case2



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial r} = (2x)\,(2r) + (2y)\,(\ln s) + (2z)\left(\frac{2r}{1+(s^2+r^2)^2}\right),$$

$$\frac{\partial w}{\partial r} = (2r^2 + 2s)\,(2r) + (2r\ln s)\ln s + (2\tan^{-1}(s^2 + r^2))\left(\frac{2r}{1 + (s^2 + r^2)^2}\right),$$

$$\frac{\partial w}{\partial r} = (4r^3 + 4rs) + (2r\ln^2 s) + \left(\frac{4r\tan^{-1}(s^2 + r^2)}{1 + (s^2 + r^2)^2}\right)$$

Now,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s},$$

$$\frac{\partial w}{\partial s} = (2x)\,(1) + (2y)\left(\frac{r}{s}\right) + (2z)\left(\frac{2s}{1 + (s^2 + r^2)^2}\right)$$

$$\frac{\partial w}{\partial s} = 2\,(r^2 + s) + (2r\ln s)\frac{r}{s} + (2\tan^{-1}(s^2 + r^2))\left(\frac{2s}{1 + (s^2 + r^2)^2}\right)$$

$$\frac{\partial w}{\partial s} = (2r^2 + 2s) + \frac{2r^2\ln s}{s} + \left(\frac{4s\tan^{-1}(s^2 + r^2)}{1 + (s^2 + r^2)^2}\right)$$

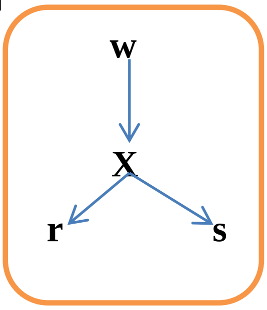
Case3: If $w = f(x)$ and $x = g(r, s)$, then w has partial derivatives with respect to r and s given by the formulas:

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

Example: If $w = f(x)$, *and* $x = r - s$, then show that $\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial s} = -[f'(x)]^2$

Solution: By using the formulas in case3



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}(1) = [f'(x)]$$

Thus, $\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial s} = -[f'(x)]^2$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}(-1) = -f'(x)$$

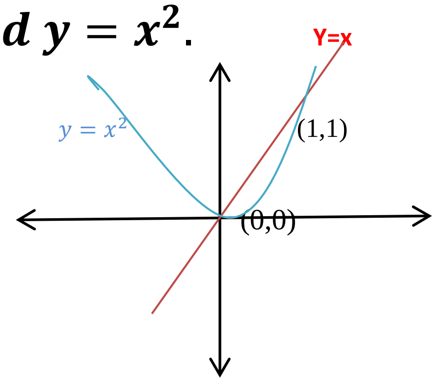
Using Green's Theorem to Evaluate line integral

If $\boldsymbol{M(x,y)}$ **and** $\boldsymbol{N(x,y)}$ **having continuous first partial derivatives in an open region containing** $\boldsymbol{R}$**. Then**

$$\oint_C \boldsymbol{M(x,y)dx + N(x,y)dy} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \boldsymbol{dA}$$

Where C is a positively simple closed curves

<u>Example</u>**: Verify Green's Theorem for** $\oint_C \boldsymbol{y^2dx + x^2dy}$**, where** $\boldsymbol{C}$ **is the closed curve of the region bounded by** $\boldsymbol{y = x \ and \ y = x^2}$**.**



Solution :

$$I = I_1 + I_2$$

Path one (I_1): when $y = x^2 \ \rightarrow dy = 2xdx$ so,

$$\oint_C \boldsymbol{y^2dx + x^2dy} \Longrightarrow \int_0^1 x^4dx + 2x^2dx = \frac{x^5}{5} + \frac{x^4}{2} \ |_0^1 = \frac{7}{10}$$

Path two (I_2): when $y = x \ \rightarrow dy = xdx$ so,

$$\oint_C \boldsymbol{y^2dx + x^2dy} \Longrightarrow \int_1^0 x^2dx + x^2dx = \frac{2x^3}{3} \ |_1^0 = \frac{-2}{3}$$

Thus, $\oint_C \boldsymbol{y^2dx + x^2dy} = \frac{7}{10} - \frac{2}{3} = \frac{1}{30}$

Now, we can use **Green's Theorem** to change the line integral into a double integral over the region R:

Taking: $\boldsymbol{M(x,y) = y^2 and \ N(x,y) = x^2}$

$$\frac{\partial M(x,y)}{\partial y} = 2y \quad and \ \frac{\partial N(x,y)}{\partial x} = 2x$$

$$\int_0^1 \int_{x^2}^{x} (2x - 2y)dydx = \int_0^1 (2xy - y^2) \ |_{x^2}^{x} dx = \int_0^1 x^2 - 2x^3 - x^4 \, dx = \frac{1}{30}$$

$$\oint_C \boldsymbol{y^2dx + x^2dy} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \boldsymbol{dA} = \frac{1}{30}$$

Example: **Verify Green's Theorem for** $\oint_C \ x^2dx + xydy$**, where** C **is the triangle having vertices** $(0,0), (1,0), and\ (1,1)$.

Solution: H.W

Exercises

Q1) Verify Green's Theorem for $\oint_C \ x^2dx + xydy$**, where** C **is the triangle having vertices** $(0,2), (2,0), and\ (4,2)$.

Q2) Verify Green's Theorem for $\oint_C \ 2xy^3dx + 4x^2y^2dy$ **, Where** C **is the closed curve of the region bounded by:** $x - axis\ , x = 1\ and\ y = x^3$.

Q3) Verify Green's Theorem for $\oint_C \ \cos y\, dx + e^x dy$

Where C **is the triangle having vertices** $(0,0), (\pi,0), and\ (\pi,\pi)$.

Q4) Verify Green's Theorem for $\oint_C \ xydx + 2dy$ **Where** C **is the closed curve of the region bounded by:** $x - axis\ and\ y = \sqrt{4 - x^2}$.

Q5) Apply Green's Theorem for $\oint_C \ xydy - y^2dx$ **where** C **is the square cut from the first quadrant by** $x = 1\ and\ y = 1$.